THE MATHEMATICS TEACHER

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EDITORIAL.

Getting our Bearings

As the sailor must get his bearings in order to know the direction to steer his craft, so the teacher must get his if he would accomplish the purposes of his high calling. The tendency is to forget our true aim in teaching and make the subject we teach an end in itself rather than a means to an end. This is apparent as one reads or listens to discussions on what to teach and why and how to teach it in both secondary school and college subjects.

The state or community supports its schools to make good citizens, and good citizenship implies good character and a trained mind. We cannot have good characters or trained minds without a developed will—a will to make us (1) attend, concentrate, etc.; (2) choose the right rather than the wrong. It appears evident, therefore, that one very important element in our educational work is the development of the will, yet this fact seldom stands out as a conscious effort on the part of the teacher.

If people would shift their viewpoint somewhat, so as to keep more prominently in mind as the primary object of education, the making of good citizens with the elements that it implies, instead of making a knowledge of the subject the primary object there would be fewer of the present day remedies for our educational ills.

What a student gets out of a subject depends to a very large extent on his attitude of mind towards it, so that interest is an

important element that must be acquired if not already present; and here lies one of the greatest difficulties in connection with the teaching of mathematics. How are we to get the students interested? Their interest is largely in proportion to the direct use they can see for the facts of the subject, and they have much assistance from the general public to lead them to this attitude of mind. The teacher will do well to do all he can to get the student to see the direct bearing of his subject on everyday life, not only from the self-evident standpoint of using the facts, but particularly from the standpoint of using the training the subject furnishes the occasion for developing.

In other words we as teachers must keep more constantly and prominently in mind the fundamental objects for which the schools are supported, viz., character and training and then see the full value of our subjects as means to this end rather than ends in themselves. Having done this we should stand up for our subject fearlessly and without apology knowing that if properly taught it will prove an instrument of the highest value for attaining the desired object.

ARITHMETIC IN HIGH SCHOOLS.*

By David Eugene Smith.

Professor Smith said that it was not his purpose to advance a thesis upon the subject, but rather to speak of the pros and cons of the question. He called attention to the difficulty of deciding such a question for all types of high schools. In particular, it is evident that schools which are almost exclusively preparatory institutions for colleges have not the time to devote to such a subject. On the other hand, commercial, or technical high schools must of necessity introduce certain work of this kind. Professor Smith was of the opinion that such work should not be of the older type for the reason that subjects like taxes and insurance have, as usually presented, almost no mathematical content. He felt that if arithmetic was introduced in any type of high school, it should be chiefly for the purpose of improving the student's power in pure calculation. In particular, he felt that logarithms, the slide rule, approximations, limits of error, and kindred subjects should have place along with the work in short methods of calculation. He called attention to certain courses in arithmetic given in the schools of Sweden, Switzerland, and Germany, as outlined, together with those of other countries, in Dr. E. H. Taylor's recent monograph. He expressed his opinion that the coming of the junior high school would give us a more modern course of study in mathematics, and that it was probable that arithmetic could be carried along at the same time as other subjects to better advantage than by placing it in a class by itself in the senior year.

* From a discussion on this subject at a meeting of the New York Section.

MATHEMATICS RELAYS FOR HIGH SCHOOLS.

By Ernest H. Koch, Jr., and Thomas H. McCormick.

One of the most hopeful signs for the advancement of secondary school education is reflected in the increased attention which is being given to the content and instruction of the first year in the high schools. To those who are concerned with this work it is always a source of interest and profit to note and measure the development of pupils during this formative period.

Test 1. *Addition.*

(1)	(2)	(3)	(4)	(5)
869	494838	67949574	369231	168.29
675	489614	83357111	445170	1175.59
534	429868	23736625	233068	935.48
	793597	22165242	320122	27.61
		42776566	382428	872.29
		64111731	625552	1015.88
			717165	1322.37
			441314	537.48
			514685	170.23
			761336	540.62
				115.21
				696.66
				1006.33
				653.96
				177.87
				28.53
				262.71
				691.14
				104.12
				282.93

(6) Add horizontally $369 + 284 + 795 + 306 + 155 =$

It is a well-known fact that a test given to an entering class of high-school freshmen will show it woefully lacking in any sustained arithmetic operation. It is for this reason that every high-school course in mathematics should provide for a resumption of the elementary school drill for a period extending from

five to ten weeks. The manner of presenting such a drill will depend upon the type of high-school course, the character of the work presented during the first year and also upon the attitude of the pupils and their teachers. It is considered advisable to learn the status of the class during the first day of school instead of wasting this time before the formal instruction is begun. It is suggested that you try Tests 1, 2, 3 and 4 upon your classes in order to determine their sustaining power in the four fundamental operations. An arbitrary but convenient basis for rating is to assign to each example a weight corresponding to its position in the set. Credit should not be given unless an example is completed without error. On this basis the maximum rating for six questions is 21. It is comparatively easy to arrange a set of examples so that the total operations in a particular example may be commensurate with its position in the set. The four tests may be given during one hour allowing a five-minute interval between tests.

TEST 2. *Subtraction.*

(1) 5782 (2) 46928743 (3) 356478912 (4) 46386721467
3798 35847865 99999999 34594677374

(5) From 4873925816 subtract 563872615 three times in succession.

TEST 3. *Multiplication.*

(1) 62×36, (2) 848×759, (3) 78654×3.1416, (4) 86372417×624, (5) $75637 \times 3 \times 5 \times 7$, (6) 1425 lbs. meat @ 18¾ cents.

TEST 4. *Division.*

(1) $365 \div 15$, (2) $4638 \div 412$, (3) $53896388208 \div 1147$, (4) $120 \div 135$, (5) $1.65 \div .9\frac{1}{11}$.

Carry divisions to two decimal places.

Many high-school teachers realize that the pupil's reluctance to continue arithmetic is due to the failure to provide expedient methods for maintaining interest in the subject. It is the purpose of the authors to present from their experience a simple successful device which maintains the interest in arithmetic in a most unexpected manner and at the same time provides the most desirable incentives for accuracy and speed in the fundamental operations. This device is in the nature of a contest and is called a Mathematics Relay because it was first used as an

event in an intellectual Three "R" Contest which was conducted like the usual athletic track meet.

The relay consists of four related parts presented in the following order: a continued addition, a continued subtraction, a continued multiplication, and a long division. It is performed by four boys who constitute a team. Each boy performs one of the operations and when he has finished the next boy follows him until all of the operations have been completed.

In order to have the entire class participate in a relay the class is divided into as many teams as there are multiples of four boys. The class teacher either selects or calls for volunteers who act as captains of teams. Each captain selects three other members to act with him as a team. If the classes exceed forty in number allow five members to a team in which case one boy in each team serves as an alternate.

All teams begin work simultaneously at the blackboards upon a signal from the teacher. The blackboards at the front of the room are adequate for this purpose and can be seen easily when the teacher is stationed at the rear of the room. The winning team is the one which finishes first with a correct answer. A member of a team may fail to do his assignment correctly and after two more vain attempts at a correct solution he is replaced by the alternate designated by the captain of the team. If the alternate fails in three attempts at the same assignment then that team forfeits its place in the contest. If the captain fails a new captain may be chosen and he in turn reorganizes his team.

At the beginning of the contest the captains write on the boards the subtrahend (S), and how many times it is to be used, the multipliers (M), and the divisors (D) similar to the following:

| S 98645, 5 times | M 6, 8, 9 | D 432 |

The captains then retire and send their respective adders to the blackboards and at a given signal they write two addends dictated by the teacher. Without any further direction and unaided he proceeds to form eight additional addends and finally sums the entire set of ten addends. The method of forming the additional addends is shown in the work below. The principle is to add two successive addends to form the one following, as shown below.

Addend (1)	34786	Dictated by the teacher.
Addend (2)	59301	Dictated by the teacher.
Addend (3)	94087	(1)+(2) added by the pupil.
Addend (4)	153388	(2)+(3) added by the pupil.
Addend (5)	247475	(3)+(4) added by the pupil.
Addend (6)	400863	(4)+(5) added by the pupil.
Addend (7)	648338	(5)+(6) added by the pupil.
Addend (8)	1049201	(6)+(7) added by the pupil.
Addend (9)	1697339	(7)+(8) added by the pupil.
Addend (10)	2746740	(8)+(9) added by the pupil.
Sum	7131718	added by the pupil.

A teacher's check which is not to be given to the pupil under any circumstance is to multiply the seventh addend by the number eleven.

When the teacher nods his head indicating a correct result the next boy of that team goes to the board and subtracts the subtrahend the required number of times from the previous boy's result.

Minuend	7131718	(previous sum).
Subtrahend	98645	
	7033073	first remainder.
Subtrahend	98645	
	6934428	second remainder.
Subtrahend	98645	
	6835783	third remainder.
Subtrahend	98645	
	6737138	fourth remainder.
Subtrahend	98645	
	6638493	final remainder.

When this work is correct the third boy goes to the board and does the required multiplication using the last boy's result as the multiplicand.

Multiplicand .	6638493
First multiplier	6
	39830958
Second multiplier	8
	318647664
Third multiplier	9
Product	2867828976

Finally the last boy of the team goes to the board and divides the product obtained by his team mate by the required divisor written at the top of the board.

Divisor.	Dividend.
432	2867828976
	6638493 quotient.

That team which completes the work correctly in the least time is declared the winner and the champion of the class. A series of challenges follows and with the reduction in the time limit of contests the classes are able to form a challenging team to meet similar teams from other classes. By the process of elimination an all school team is chosen and this team represents the school in interannex or interschool contests.

The development of relay teams involves from five to ten minutes' drill daily during the arithmetic class period if there is one, otherwise it may be conducted by the official teacher or section officer during a study period. It needs no further comment to say that once relays are started the pupils will take care of themselves for practice whenever they may have the privilege to use blackboards before, between and after school sessions.

Incidental to these relays are the separate drills in the four fundamental operations, each of which provides opportunities for contests. They are important in themselves and become interesting because they are contributing factors to the relays.

Addition drills are always more interesting when they involve the formation of addends from two given addends and the subsequent summing of all the addends. It is wise to begin addition with numbers of the fourth order and gradually increase their magnitude until numbers of the ninth order are used. It is also desirable to limit the number of addends to six at the beginning and then rapidly increase this number to ten addends. For general class drills in addition the following order of difficulty is suggested. The number to the right of the two given addends indicates how many addends are to be used for the sum.

375		9437		88509		76549	
246	6	8724	6	24763	6	28763	10

Drills in subtraction require that a given subtrahend shall be

used repeatedly and successively a definite number of times. It resembles a reverse of the process of addition described above excepting that the same operator is used.

Drills in multiplication involve two or more consecutive multiplications in which the multipliers are chosen from the nine digits. Variations are made by repeating the multipliers in different arrangements.

Drills in division involve a single divisor of low order. When the quotient is not integral it is required to express the remainder as a fraction in its lowest terms. Long division should be done directly without partition of the divisor. It is interesting to make the divisor the product of the multipliers of a previous multiplication example at the same time using the previous product as the new dividend.

If the teacher has a ready command of numbers he can use operators at random and obtain his answers for checking by short-cut methods.

An easy way to obviate the desire on the part of some pupils to cheat is to use a 6×9 yellow pad for drill work. After the sheets are distributed to the class each pupil is instructed to fold the paper into quarters. This provides eight spaces, each space being reserved for a single example. One space may be used for the date, name, class, time consumed in test, and the grade. Designate each of the eight spaces by number and dictate the operators for each example designating the space into which it is to be placed. When all the dictation has been made have the papers folded, exposing space one. At a given signal have the pupils blaze away but always keeping the papers folded to the space in which they are working. This makes a foolproof device which entirely eliminates opportunities for dishonest work. It should require less than a minute to distribute paper and dictate the examples. At first it will require about two minutes for each example, but this time may be cut in half after a week of drill and should reduce considerably but less rapidly after five weeks' drill. A relay such as is shown in this article should be completed within five minutes.

The following material has been used in the class rooms of the authors and will be found of great service, as the answers accompany each example. The examples in addition all involve

ten addends. The subtraction examples all involve a repetition of the subtrahend five times. The separate multipliers are indicated by the commas which separate them.

ADDITION.

34786	54732	76592	76459	17809
59301	83196	31480	95467	37211
7131718	10331508	6982800	12606341	4254063

SUBTRACTION.

7131718	10331508	6982800	12606341	4254063
98645	45678	78692	82821	54063
6638493	10103118	6589340	12192236	3983748

MULTIPLICATION.

6638493	10103118	6589340	12192236	3068994
6, 8, 9	5, 7, 9	4, 7, 6	7, 7, 7	8, 3, 5
2867828976	3182482170	1107009120	4181936948	368279280

DIVISION.

432 | 2867828976	315 | 3182482170	672 | 107009120
6638493	10103118	1647335
196 | 4181936948	144 | 368279280	
21336413	2557495	

The following examples are appended for relay work but it should be noted that the examples listed above may be combined to make relay examples. In each case the addends are to be used ten times and the subtrahend is to be repeated five times.

RELAY EXAMPLES.

47591	34758	74289	93897
24863	76828	35927	47095
4805449 sum.	8672224 sum.	7247471 sum.	9308695 sum.
78965 sub.	84628 sub.	83927 sub.	79317 sub.
4410634 rem.	8249084 rem.	6827836 rem.	8912110 rem.
7, 5, 9 mul.	4, 8, 7 mul.	7, 9, 4 mul.	8, 7, 6 mul.
1389349710 prod.	1847794816 prod.	1720614672 prod.	2994468960 prod.
630 div.	896 div.	336 div.	672 div.
2205317 quot.	2062271 quot.	5120877 quot.	4456055 quot.

The authors are not aware that the mathematics relay has

been tried elsewhere before its introduction at the High School of Commerce in New York City. It will give the authors pleasure to learn the results of trials by other teachers. This device has demonstrated, to business men at three public exhibitions, that it is quite possible to train pupils in accuracy and speed in an effective and interesting manner and so they may continue to maintain the highest respect for the technique of computation.

GEOMETRIC STEREOGRAMS—A DEVICE FOR MAKING SOLID GEOMETRY TANGIBLE TO THE AVERAGE STUDENT.

By Walter Francis Shenton.

In all the gamut of mathematical studies there is not another subject which has given teachers so much concern as Solid Geometry. Time and thought and energy to the n-th power have been given to the preparation of the subject matter, and to the revision of the course so as to make it a logical successor to Plane Geometry, but after a class has been carefully taught by the best available methods it is almost heart-rending to see how many members of the class do not have even the fundamental ideas of space relation correctly fixed in their minds.

In order to make these space relations clear the teacher has had for his resources up to the present time such models as may be bought or made for demonstration and use in the class room. Wood and plaster and silk strings, to say nothing of cards and wires have been put together in the various shapes that the theorems of solid geometry require for demonstration and from them the students have tried to understand the meaning of the plane representation of the solid figure. Of these models we have nothing to say in any unkind way, for they were the means of giving us our first complete notion of the complex relations of lines and planes in space, and more than once a desk top and a number of pencils have illustrated to the writer certain relations that he could not otherwise fully comprehend. But at the very best they are clumsy and friable and the student must be under the watchful eye of the teacher always when he is using them lest damage result to the models. But this is not their worst fault. It consists in the fact that they are not available to the individual student when he is preparing his work. Even supposing that they do help him straighten out the tangle he had gotten into during his study of the night before, how much better it would have been if he had never gotten into the tangle!

This condition of affairs has made it desirable that each student should have a set of simple models of his own which he may keep with him when he studies and thus form correct ideas from the start. To ask that this may be literally fulfilled is too evidently futile to be worthy of further consideration. However, after a fashion this very thing has been done by the firm of Underwood and Underwood whose name is common property at the present in connection with the finest photographs of current events that are published, and whose travel photographs in stereoscopic form now accord many of us the privilege of seeing in a very realistic way, scenes of countries whose soil we may never hope to tread. They have met the solid geometry need to some small degree by a series of illustrations for various solid figures which when viewed through their stereoscope give relief. Their method, however, has proved too bulky and expensive for the average student.

However, in the past quarter of a century there has been working out an old idea in new form, and it is the purpose of this paper to give briefly the history, theory, and results of this method. In 1853 there appeared in the *Poggendorf Annalen*, an article by W. Rollman (*90*, p. 187) in which was set forth a method of using a two-colored figure viewed with a pair of two colored glasses to obtain the effect of relief. D'Almeida used this method in 1858—probably independently re-discovered it—in the form of viewing with bi-colored glasses a two-colored figure thrown on the screen by a lantern and this same method was also developed by Molteni. However, it did not come before the public eye to any considerable extent until 1894 when L. D. Duhauron had a modification of this method patented in this and other countries. His method took cognizance of the work of the previous inventors and was based entirely on the idea of producing actual drawings, half-tones, or any other forms of actual print, which were to be viewed with a pair of colored glasses as will be explained shortly. Although he produced quite a collection of prints marketed as "Plastographical Views of the World," the method did not seem to have made a serious impression upon the mathematical world until the Cambridge International Congress of Mathematicians in 1912, when M. H. Richard and M. H. Vuibert presented a set of "Ana-

glyphes" as they called them. The figures showed applications to ordinary solid geometry, spherical geometry, descriptive geometry, crystallography and physics, and were very well received. In the fall of the same year they published a little brochure entitled "Les Anaglyphes Geometriques" which described the theory of the process and gave some twelve pages of figures which when viewed through the glasses which accompanied the text gave plastographic effect.

This little book set the writer seriously to work as he had been much interested in the making of models and this seemed a very practical way out of the general difficulty. The results of the work of the past three years are presented briefly in what follows.

The physiological effect of seeing a solid figure in space is a comparatively simple thing. Each eye, acting as a small camera, sees a small reproduction of the body in front of it and within its range of vision. This reproduction is essentially a section of the pencil of rays passing through a focal point somewhere near the pupil of the eye. As the two pupils are at a distance of about seven centimeters from each other, each eye sees the figure from a different point of view and so each of the reproductions is a section of a different pencil of rays. If now, a cube, say, be set on a sheet of paper and viewed from a fixed position of the eyes and the eyes closed alternately while the section of the ray-pencil is sketched on the paper for each eye, we will have a peculiar double trace of the figure, one half of which corresponds to the section of the pencil of rays for each eye by the plane of the sheet of paper. Since each section of this pencil will give the same effect to the eye concerned as the original solid, if we can in any way separate the two figures so that each eye sees only the section that belongs to it, we shall have the same physiological effect as would be caused by the actual solid.

This may be accomplished by calling into play the following physical phenomenon. If a line of pure color be drawn on a sheet of white paper and then viewed through a plane screen of the same color, that is, which transmits rays of light having the same frequency as those given off by the colored line, the line will apparently disappear. What actually happens is that the

whole white sheet appears through the screen to be of exactly the same color as the line, which is then lost in the common tone of the background. If now we view a line of contrasting color through the screen, it will not reflect any rays of the frequency that will pass through the screen, and so will appear through it as a black line on a colored ground.

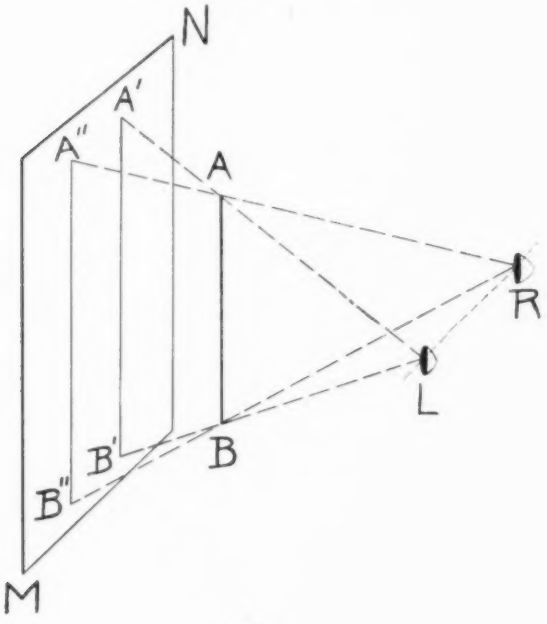


FIG. 1.

Returning now to our original proposition. If we were to draw on a sheet of white paper in green, for example, the section of a figure corresponding to what the right eye would see, and in complementary red the section corresponding to that for the left eye, then each eye would see its own figure in black on a colored ground, provided that the right eye were screened with red and the left eye with green. By choosing the red and green as nearly complementary as may be in practice, the effect to the two eyes due to the mixing of the two backgrounds will be a grayish white, and the effect of the two black figures will be that of a solid in relief. This is the physical theory of the method.

Let us now turn our attention to the methods at hand for making the drawings, which methods are in practice somewhat hard to develop. First consider the very simplest case, when the object which we wish to draw, say a vertical straight line in space, lies between the observer and a plane which is vertical to the earth's surface and parallel to the line joining the pupils of the eyes of the observer. In this case the two figures desired are the two traces made in the given plane by the two planes which are determined by the given line and the pupil of one eye of the observer. In Fig. 1 suppose that AB is the given vertical line in space and that MN is the vertical background plane, while the eyes are denoted by their initial letters L and R. Then $A'B'$, the intersection of the triangle LAB with MN, is the line in our figure that corresponds to the left eye; and $A''B''$, the intersection of the triangle RAB with MN, is the line which corresponds to the right eye. However, this requires that we have the actual line in space, and we have actually used this method more or less extensively in preparing some figures when we had the actual models at hand in convenient shape and size. When the figure becomes too complex, or no models are at hand, we must use an extension of this method. In the designing of figures we make use of the three orthogonal projections of the solid in the three planes of reference, from which, by devices that are beyond the scope of this paper we are always able to produce a correctly drawn figure without even having visualized the solid.

The method just outlined is such as we would have to use for figures which are to be projected on a vertical screen by means of such apparatus as a reflectoscope, and viewed by a class, or for figures which we wish held vertically before the observer. A rather more natural way to produce the drawings is to make them appear to rise from a sheet of paper laid horizontally on a desk and viewed so that the line of sight makes an angle of say 45° with the vertical. As this causes the figures to have the actual appearance of standing up on the sheet it produces a very startling effect. The general scheme for producing the figures of a vertical line viewed in this way is shown in Fig. 2.

One of the very interesting things in connection with drawings made by the first method is that if the figures are not too com-

plex, when we view them with the glasses reversed they appear to penetrate the paper instead of coming out from it as is normal. By this means what would normally appear as a frustum of a cone standing out from the sheet, its base on the background, becomes, when the glasses are reversed, a much longer frustum hollowed out of the sheet. Many persons have difficulty in seeing this because they cannot persuade themselves that the figure penetrates the paper as it seems to do in this case.

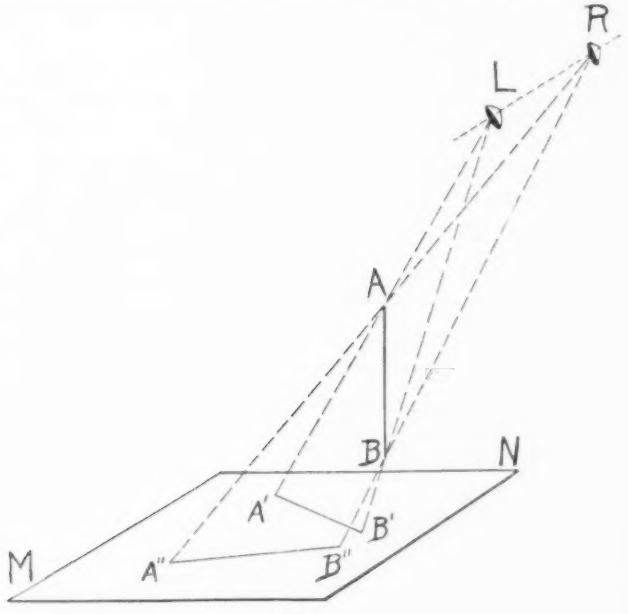


FIG. 2.

It has been a rather interesting observation that almost any school boy will be able to raise the figures at once and describe exactly what he sees there, while some of the more perfectly trained mathematicians of my acquaintance have difficulty in making the figures seem real. In any instance where there is difficulty in seeing the figure clearly in relief, it can almost always be overcome by asking the spectator to move his head from side to side, or else to move the drawing slowly from right to left and back, in which case the figures will appear to

rise from the paper and follow his eyes as they move. This peculiar moving of the figures is one of the interesting things in connection with these drawings.

Quite a number of technical difficulties have been met in trying to make a commercial proposition out of this theory. Paramount among them has been the difficulty of getting inks and screens for the drawings, which will behave as we have indicated is necessary. Having overcome this obstacle by dyeing our own screens with the same dyes that we use for our drawing inks, we were face to face with a more serious trouble in the nature of being unable to get printer's inks which would work as we desired. After about two years of experimenting along this line we have been able to produce prints which are even better than our original drawings. In addition we have been able to find a limited supply of gelatine screens of the colors desired, so that the problem seems to have been solved, technically speaking.

It is our purpose to produce a set of drawings to cover all the figures of solid geometry, lettering them to agree with one or several of the texts now in most common use. The larger number of these will be drawings like the sample that we have submitted here, but such figures as the sphere and other curvilinear solids will probably be presented by means of half-tones from photographs of actual models.

We believe that figures presented in this way will be an invaluable addition to the teaching equipment now accessible for solid geometry and that they will stimulate the students to a better understanding of one of the dark places in mathematics. This method possesses the following very distinct advantages over any other similar method that has been suggested:

1. It is just as easy on the eyes as looking at the actual space figures would be.

2. It is self-contained. A small book or portfolio with an envelope in the cover to contain the glasses would be the sum total of apparatus required.

3. The simplicity of the apparatus will make the process inexpensive. So inexpensive, we believe, that each student can afford to have a set of his very own.

4. The figures with their bold relief will immediately fascinate

the average student, especially when he discovers that he can make them bend for him when he moves his head from side to side. I have seen this very vividly illustrated by the delight that both teachers and students have shown when they made this latter discovery.

5. An actual model bearing the same letters as the black-and-white plane figure in the text, used in the student's room, will not only enable him to understand the proposition, but will also make it possible for him to understand the "stupid" figure in the text.

Because of these advantages we claim that we have here a novelty which is destined to play a very large part in the teaching of solid geometry to the average and below-average student.

Note.—A limited supply of prints illustrating the proposition, *The volume of any parallelopiped is equal to the product of its base and altitude*, together with colored glasses, may be obtained at the rate of twenty-five cents for one print with pair of glasses. Address,

WALTER F. SHENTON,
Johns Hopkins University,
Baltimore, Md.

JOHNS HOPKINS UNIVERSITY,
BALTIMORE, MD., April 1, 1915.

DECOMPOSITION INTO PARTIAL FRACTIONS.

By Eugene Randolph Smith.

In my former article on this subject in The Mathematics Teacher of June, 1914, I did not bring out one point, which, on further consideration, seems to me to be the key to this process. Dr. G. F. Metzler's excellent article in the issue of June, 1915, further emphasizes this need, for while he uses the point of which I speak in various ways, he does not bring the whole question to this focus.

Without elaboration, the fact is as follows: In the fraction $N/D \cdot F^k$ (where F is any prime factor of the denominator, D is the co-factor prime to F, and K is the positive integral power to which F appears) the numerators for the partial fractions of denominators F^k, F^{k-1}, $\cdots F$ are the successive coefficients in the quotient obtained from the detached coefficient function of F, $(R_1 + R_2 + \cdots R_k)/(r_1 + r_2 + \cdots r_k)$, R_1 to R_k being successive remainders obtained by dividing N by F, and r_1 to r_k being successive remainders obtained by dividing D by F.

It is evident that this reduces the method in partial fraction to

(1) Dividing the numerator and the co-factor in the denominator by the factor in question, repeating the division into each successive quotient until the number of divisions equals the number of partial fractions to be obtained.

(2) Dividing the function having the numerator remainders as coefficients by the function having the denominator remainders as coefficients.

(3) Simplifying the coefficients of the quotient (when necessary) and using them as the required numerators.

Illustrations.

(For the convenience of those who may wish to compare the methods, many of the examples used by Dr. Metzler are also used here.)

A. Involving the First Power of a Function of First Degree.

To obtain the numerator for $x+1$ from the fraction

$$x^2+1/[(x-1)^4(x^3+1)].$$

(No. 4 in Dr. Metzler's article.)

Divide* x^2+1 and $(x-1)^4(x^2-x+1)$ by $x+1$.

Numerator.		*Denominator.*	
	$1+0+1$		
-1	$1-1+2$	$(-2)^4\cdot 3$	$=48.$

$$\therefore \quad \frac{R_1}{r_1}=\frac{2}{48}=\frac{1}{24}$$

and the result is

$$\frac{1}{24(x+1)}.$$

B. Involving a Function of the First Degree Used to a Power Higher than the First.

To obtain the numerators for $(x-1)^4$, $(x-1)^3$, $\cdots$ $(x-1)$ in the same fraction.

Divide x^2+1 and x^3+1 by $x-1$ four times.

Numerator.		*Denominator.*	
	$1+0+1$		$1+0+0+1$
	$1+1+2=R_1$		$1+1+1+2=r_1$
1	$1+2=R_2$	1	$1+2+3=r_2$
	$1=R_3$		$1+3=r_3$
$0=R_4$			$1=r_4$

Take the quotient of these remainder functions

	$2+2+1+0$
-1	$\quad -1$
-3	$\quad -3+\frac{3}{2}$
-3	$-3+\frac{3}{2}+\frac{3}{4}$
2	$1-\frac{1}{2}-\frac{1}{4}+\frac{5}{8}$

* The substitution of $x=-1$ from $x+1=0$ is an identical operation in such a case, and may be used if desired. It is simpler in the denominator of this example.

giving

$$\frac{1}{(x-1)^4} - \frac{1}{2(x-1)^3} - \frac{1}{4(x-1)^2} - \frac{5}{8(x-1)}.$$

In A and B the remainders were of necessity independent of x, and the results were therefore in simplest form. In the types to follow this is not necessarily true, for when the remainders $r_1 \cdots r_k$ are functions of x, the results obtained will involve x in the denominator and will therefore need further simplification. This possibility occurs when F is of degree higher than the first.

C. To Obtain the Numerator for x^2-x+1 in the Same Fraction.

Divide x^2+1 and $(x-1)^4(x+1)$ by x^2+x+1.

Numerator.

	$1+0+1$	
-1	$\quad -1$	
1	$\quad 1$	
1	$1 \quad 1 \quad 0 = R_1$	

Denominator.

	$1-3+2+2-3+1$	
-1	$\quad -1 \quad 2 \quad 1-3$	
1	$\quad 1-2-1+3$	
1	$1-2-1+3+1-2 = r_1$	

Then $R_1/r_1 = x/(x-2)$ which is not in simplest form.

The problem is therefore to transform $x/(x-2)$ into a fraction with numerical denominator, using the condition $x^2-x+1=0$.

Method I. By Auxiliary Factor.—Divide x^2-x+1 by $x-2$, obtaining a quotient $x+1$ and a remainder that is not to be used.

Multiply numerator and denominator of $x/(x-2)$ by $x+1$, giving $(x^2+x)/(x^2-x-2)$. Substitute $x^2=x-1$ from $x^2-x+1=0$, giving $(2x-1)/-3$ and the required partial fraction is $-(2x-1)/3(x^2-x+1)$.

The auxiliary factor can also be obtained by formula. For a fraction of denominator $mx+n$, when $ax^2+bx+c=0$, it is

$$\left[amx - \begin{vmatrix} m & n \\ a & b \end{vmatrix}\right].$$

This can be remembered by the fact that the first coefficient is the product of the two given first coefficients, while the second

coefficient is the determinant of the first and second coefficients in the two given functions.

For example, to reduce $(3x-2)/(5x-7)$ to a fraction of numerical denominator when $2x^2-7x+1=0$.

The auxiliary factor is

$$\left\{ 2\cdot 5x - \begin{vmatrix} 5 & -7 \\ 2 & -7 \end{vmatrix} \right\} = 10x - 21.$$

$$\frac{(3x-2)(10x-21)}{(5x-7)(10x-21)} = \frac{30x^2-83x+42}{50x^2-175x+147}.$$

Substituting $2x^2=7x-1$, this reduces to $(22x+27)/122$.

It would be quite possible to reduce such a fraction by substituting in it the value of x obtained from the quadratic, but that, in general, is a more difficult process.

Method II. By Formula.—The fraction $(kx+l)/(mx+n)$ when $ax^2+bx+c=0$, reduces to

$$\frac{a\begin{vmatrix} m & n \\ k & l \end{vmatrix} x - \begin{vmatrix} a & b & c \\ m & n & 0 \\ 0 & k & l \end{vmatrix}}{-\begin{vmatrix} a & b & c \\ m & n & 0 \\ 0 & m & n \end{vmatrix}}.$$

This arrangement of coefficients is easy to remember, and the determinant evaluates simply in terms of the last column.

Using the same example, $(3x-2)/(5x-7)$ when

$$2x^2-7x+1=0$$

the result is

$$\frac{2\begin{vmatrix} 5 & -7 \\ 3 & -2 \end{vmatrix} x - \begin{vmatrix} 2 & -7 & 1 \\ 5 & -7 & 0 \\ 0 & 3 & -2 \end{vmatrix}}{-\begin{vmatrix} 2 & -7 & 1 \\ 5 & -7 & 0 \\ 0 & 5 & -7 \end{vmatrix}} = \frac{22x+27}{122}.$$

Method III. By Undetermined Coefficients.—Let

$$x/(x-2) = ax + b.$$

Then $x = ax^2 + (b-2a)x - 2b$, but since $x^2 - x + 1 = 0$ $ax^2 = ax - a$, giving $(b-a)x - (a+2b) = x$.

Therefore $b - a = 1$

$$a + 2b = 0 \quad \text{and} \quad b = 1/3, \quad a = -2/3$$

giving, $(-2x+1)/3$.

Method IV. By Partial Fractions.—In $x/[(x-2)(x^2-x+1)]$, substituting $x = 2$, the numerator for $x-2$ would be $\frac{2}{3}$; therefore subtracting $2/[3(x-2)]$ from this fraction, we obtain

$$\frac{3x - 2(x^2 - x + 1)}{3(x-2)(x^2-x+1)} = \frac{-2x^2 + 5x - 2}{3(x-2)(x^2-x+1)} = \frac{-2x+1}{3(x^2-x+1)}.$$

D. The only case not yet considered is that of an unfactorable denominator that occurs to some power, as in $N/[D(ax^2 + bx + c)^k]$. The method is again that of obtaining k remainders by dividing N and D by $ax^2 + bx + c$ in successive divisions, and taking the quotient of the functions obtained to k terms. The coefficients evaluated for $ax^2 + bx + c = 0$ are the respective numerators. In $(x^2+2)/[(x-2)^2(x^2+1)^4(x^2-x+1)^2]$ (No. 5 in Dr. Metzler's article), to find the numerators for $(x^2-x+1)^2$, etc., and for $(x^2+1)^4$, etc.

Using $x^2 - x + 1$ as a divisor,

	Numerator.	*Denominator.*
	$1 + 0 + 2$	$(x-2)^2(x^2+1)^4 \div (x^2 - x + 1)$ gives
-1	$\quad -1$	$r_1 = -3, \quad r_2 = 14x - 6.$
1	$\quad 1$	
1	$1 \quad 1 + 1 = R_1$	
	$1 = R_2$	

The solution of the detached coefficient identity

$$\frac{(x+1) + (1)}{-3 + (14x - 6)} = (ax + b) + (cx + d)$$

will give the required numerators.

$$\begin{array}{c|l} & (1+1)+(1) \\ \hline -14+6 & \dfrac{22-20}{3}+\dfrac{*14}{3} \\ \hline -3 & -\dfrac{1+1}{3}-\dfrac{22-17}{9} \end{array}$$

giving

$$-\frac{x+1}{3(x^2-x+1)^2}-\frac{22x-17}{9(x^2-x+1)}$$

as the required fractions. In this case since r_1 is independent of x, the results are in simplest form without further change.

Using x^2+1 as a divisor,

$R_1=1,\quad R_2=1,\quad R_3=0,\quad R_4=0,$

$r_1=4x-3,\quad r_2=-10x-6,\quad r_3=-6x+12,\quad r_4=1.$

Then the required numerators can be obtained from

$$\begin{array}{r} (1)-(1) \\ (4x-3)+(-10x-6)+(-6x+12)+1 \end{array}$$

$$=(ax+b)+(cx+d)+(ex+f)+(gx+h).$$

It will be observed in the following division that besides the terms of the divisor there is used another term, -4. This term and its products are marked. This is the first term of the divisor $4x-3$ used with sign changed, to correct the error which would otherwise be introduced by the following fact. As can be seen by clearing of fractions in the preceding identity, each multiplication of a term in the right hand member by $4x-3$ gives a second degree product, and therefore gives a multiple of x^2+1

* The product $-\left(\frac{1+1}{3}\right)(-14+6)$ contains $\frac{14(x^2-x+1)}{3}+\frac{22x-20}{3}$

The $14(x^2-x+1)/3$ belongs in the next column as it is the next higher power of the factor x^2-x+1, in terms of which the identity is being considered. If three numerators were to be obtained, this term would need to be added in with the third column. This will be more evident in the process of finding the numerator for the powers of x^2+1, where the numbers carried forward in this way will also be indicated by the sign * and will be placed directly following the product from which they are obtained.

which should therefore be transferred to the next column. The starred terms correct the similar error for all terms of the divisor except the first. Note that only the x coefficient of the preceding term of the quotient should be multiplied by this number.

	$(0+1)$	$+(0+1)$	$+(0+0)$	$+(0+0)$
-4° (with x coefficient only)		$\frac{16^{\circ}}{25}$	$\frac{72^{\circ}}{125}$	$-\frac{2272^{\circ}}{625}$
$0-1$				
$6-12$			$\frac{30+60}{25}$	$-\frac{{}^{*}24}{25}\ \frac{-1350+5400}{625}$
$10+6$		$\frac{-54+22}{25}$	$-\frac{{}^{*}40}{25}\ \frac{-918-306}{125}$	$\frac{{}^{*}180}{125}\ \frac{-1932-3204}{625}$
$4-3$	$-\frac{4+3}{25}$	$\frac{(54-63)(4+3)}{625}$	$\frac{(768+134)(4+3)}{3125}$	$\frac{(3182+7181)(4+3)}{25^3}$

or, inverted,*

$$-\frac{4+3}{25}$$

using $x^2+1=0$

$$-\frac{4+3}{25}\ -\frac{18x+81}{125}\ \frac{568x-534}{625}\ \frac{7654x+1763}{3125}$$

giving the fractions

$$-\frac{(4x+3)}{25(x^2+1)^4}-\frac{18x+81}{125(x^2+1)^3}+\frac{568x-534}{625(x^2+1)^2}+\frac{7654x+1763}{3125(x^2+1)}$$

as a result.

While there may be some doubt as to whether this is the shortest possible process in all cases, numerous trials have seemed to show that it is the quickest method to apply in the usual examples. At any rate it has the advantage of being general in its application. An important special case is that when $r_1=1$, r_2 . . . being zero, then R_1, R_2, etc., are the required results.†

†Since to divide by $4x-3$ is to multiply by $1/(4x-3)$ or $-(4x+3)/25$ (using $x^2+1=0$), it is simpler to use this form throughout, thus saving the duplication of this process on each denominator.

Special Case Simplification.

Often in questions having two factors in the denominator, and sometimes in other cases, it is easier to subtract the simpler partial fractions from the given fraction, and then to obtain the other numerators by the easy method of perfect powers.†

In

$$\frac{x^2 - x + 1}{(x^2 - x + 2)^2(x^2 + x + 3)^3}$$

(No. 6 in Dr. Metzler's article).

Find the fractions

$$\frac{29 + 10x}{1331(x^2 - x + 2)^2} + \frac{22x - 60}{1331(x^2 - x + 2)}$$

by the usual method.

Subtracting these fractions from the given fraction leaves a numerator $1/1331(-22x^9 + 16x^8 - 132x^7 + 315x^6 - 329x^5 + 1284x^4 - 755x^3 + 3743x^2 - 1952x + 3788)$:

Divide this twice by $x^2 - x + 2$ (evenly) then three times by $x^2 + x + 3$, leaving

$$R_1 = \frac{1}{1331}(242x + 242), \quad R_2 = \frac{1}{1331}(44x - 187),$$

$$R_3 = \frac{1}{1331}(-22x + 16).$$

The results

$$\frac{242x + 242}{1331(x^2 + x + 3)^3} - \frac{44x + 187}{1331(x^2 + x + 3)^2} - \frac{22x - 16}{1331(x^2 + x + 3)}$$

are obtained.

THE USE OF DETERMINANTS.

An interesting variation in the use of remainders to solve partial fractions, is the expression of the required results in determinants whose constituents are the remainders.

The numerators for successive descending powers of a first degree factor of the denominator, if $R_1, R_2, \ldots, R_n$ are the successive remainders obtained by dividing the numerator by

† Both of these facts are used in Chap. XVIV. of "A New Algebra," by Barnard and Childs.

that factor, and $r_1, r_2, \ldots, r_n$ are similar remainders from the denominator (exclusive of the factor in question), are

$$\frac{R_1}{r_1}; \quad -\frac{\begin{vmatrix} R_1 & R_2 \\ r_1 & r_2 \end{vmatrix}}{r_1^2}; \quad \frac{\begin{vmatrix} R_1 & R_2 & R_3 \\ r_1 & r_2 & r_3 \\ 0 & r_1 & r_2 \end{vmatrix}}{r_1^3};$$

and, for the nth numerator,

$$\frac{(-1)^{n+1}}{r_1^n}\begin{vmatrix} R_1 & R_2 & \cdots & R_n \\ r_1 & r_2 & \cdots & r_n \\ 0 & r_1 & \cdots & r_{n-1} \\ 0 & 0 & r_1 \cdots & r_{n-2} \\ \cdot & \cdot & \cdot \cdot \cdot & \cdot \\ 0 & 0 & 0 \cdots & r_2 \end{vmatrix}.$$

The evaluation of these determinants is very simple, if it is done by multiplying the elements of the last column by their minors. This reduces each new determinant to the ones of lower order already evaluated.

$$\frac{9x^5 - 62x^4 + 171x^3 - 257x^2 + 239x - 90}{(x-2)^4(x^2+x+2)}.$$

Dividing by $x - 2$,

$$\begin{vmatrix} R_1 = 24, & R_2 = -1, & R_3 = 1, & R_4 = 35 \\ r_1 = 8, & r_2 = 5, & r_3 = 1, & r_4 = 0 \end{vmatrix}$$

$$\frac{R_1}{r_1} = 3; \quad -\frac{\begin{vmatrix} R_1 & R_2 \\ r_1 & r_2 \end{vmatrix}}{r_1^2} = -\frac{\begin{vmatrix} 24 & -1 \\ 8 & 5 \end{vmatrix}}{8} = -2;$$

$$\frac{\begin{vmatrix} 24 & -1 & 1 \\ 8 & 5 & 1 \\ 0 & 8 & 5 \end{vmatrix}}{8^3} = \frac{5\begin{vmatrix} 24 & -1 \\ 8 & 5 \end{vmatrix} - 1\begin{vmatrix} 24 & -1 \\ 0 & 8 \end{vmatrix} + 1\begin{vmatrix} 8 & 5 \\ 0 & 8 \end{vmatrix}}{8^3}$$

$$= \frac{5 \cdot 128 - 1 \cdot 24 \cdot 8 + 1 \cdot 8 \cdot 8}{8^3} = \frac{10 - 3 + 1}{8} = 1.$$

Similarly

$$-\frac{\begin{vmatrix} 24 & -1 & 1 & 35 \\ 8 & 5 & 1 & 0 \\ 0 & 8 & 5 & 1 \\ 0 & 0 & 8 & 5 \end{vmatrix}}{8^4}$$

evaluates in terms of the previously used determinants, giving 4.

The solution is therefore

$$\frac{3}{(x-2)^4} - \frac{2}{(x-2)^3} + \frac{1}{(x-2)^2} + \frac{4}{(x-2)}.$$

When powers of second degree prime factors are used, the coefficients of the two term remainders are used, and the primed remainders gotten by carrying the division one term further in the denominator are also used.

If the remainders in the numerator are

$$S_1x + R_1; \quad S_2x + R_2; \quad S_3x + R_3, \text{ etc.},$$

and in the denominator are

$$s_1x + r_1; \quad s_2x + r_2; \quad s_3x + r_3, \text{ etc.},$$

with the remainders for the denominator co-factor times x

$$s_1'x + r_1'; \quad s_2'x + r_2', \text{ etc.},$$

the successive numerators are, when

$$\Delta = \begin{vmatrix} s_1 & r_1 \\ s_1' & r_1' \end{vmatrix},$$

$$\frac{-x\begin{vmatrix} S_1 & R_1 \\ s_1 & r_1 \end{vmatrix} + \begin{vmatrix} S_1 & R_1 \\ s_1' & r_1' \end{vmatrix}}{\Delta};$$

$$\frac{-x\begin{vmatrix} S_1 & R_2 & S_2 & R_2 \\ s_1 & r_1 & s_2 & r_2 \\ s_1' & r_1' & s_2' & r_2' \\ 0 & 0 & s_1 & r_1 \end{vmatrix} + \begin{vmatrix} S_1 & R_1 & S_2 & R_2 \\ s_1 & r_1 & s_2 & r_2 \\ s_1' & r_1' & s_2' & r_2' \\ 0 & 0 & s_1' & r_1' \end{vmatrix}}{\Delta^2};$$

$$-x\,\frac{\begin{vmatrix} S_1 & R_1 & S_2 & R_2 & S_3 & R_3 \\ s_1 & r_1 & s_2 & r_2 & s_3 & r_3 \\ s_1' & r_1' & s_2' & r_2' & s_3' & r_3' \\ 0 & 0 & s_1 & r_1 & s_2 & r_2 \\ 0 & 0 & s_1' & r_1' & s_2' & r_2' \\ 0 & 0 & 0 & 0 & s_1 & r_1 \end{vmatrix} + \begin{vmatrix} S_1 & R_1 & S_2 & R_2 & S_3 & R_3 \\ s_1 & s_1 & s_2 & r_2 & s_3 & r_3 \\ s_1' & r_1' & s_2' & r_2' & s_3' & r_3' \\ 0 & 0 & s_1 & r_1 & s_2 & r_2 \\ 0 & 0 & s_1' & r_1' & s_2' & r_2' \\ 0 & 0 & 0 & 0 & s_1' & r_1' \end{vmatrix}}{\Delta^3}\text{; etc.}$$

It will be seen that in this case the determinants should be evaluated in terms of the minors of the second order in the last two columns.

Using $$\frac{x^2+2}{(x-2)^2(x^2+1)^4(x-x+1)^2}$$

(No. 5 in Dr. Metzler's article).

To find the numerators for successive powers of x^2+1, use x^2+1 as a divisor. The numerator remainders are seen by inspection to be 1, 1, 0, 0.

In the denominator, taking $(x-2)^2(x^2-x+1)^2$ as the dividend the following remainders are obtained.

$$\begin{array}{r|l} & 1-6+15-22+21-12+4 \\ -1 & \quad -1+6-14+16-7-4 \\ 0 & \\ \hline 1 & 1-6+14-16+7+4-3 = s_1+r_1 \\ -1 & \quad -1+6-13+10-3-4 = s_1'+r_1' \\ 0 & \\ \hline 1 & 1-6+13-10-6 = s_2+r_2 \\ -1 & \quad -1+6-6+14 = s_2'+r_2' \\ 0 & \\ \hline 1 & 1-6+12 = s_3+r_3 \\ & \quad 12-4 = s_3'+r_3' \end{array}$$

$1 = r_4$,

$1-6 = s_4'+r_4'$.

Then the first numerator equals

$$\frac{-x\begin{vmatrix} 0 & 1 \\ 4 & -3 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ -3 & 4 \end{vmatrix}}{\Delta},$$

when

$$\Delta = \begin{vmatrix} 4 & -3 \\ -3 & -4 \end{vmatrix} = -25,$$

giving

$$\frac{+4x+3}{-25} = -\frac{4x+3}{25}.$$

The second numerator equals

$$\frac{-x\begin{vmatrix} 0 & 1 & 0 & 1 \\ 4 & -3 & -10 & -6 \\ -3 & -4 & -6 & 14 \\ 0 & 0 & 4 & -3 \end{vmatrix} + \begin{vmatrix} 0 & 1 & 0 & 1 \\ 4 & -3 & -10 & -6 \\ -3 & -4 & -6 & 14 \\ 0 & 0 & -3 & -4 \end{vmatrix}}{(-25)^2}$$

$$= \frac{-90x-405}{25} = -\frac{18x+81}{125}.$$

In evaluating these determinants but three products are used: in these products all the minors of the first two columns are known from the preceding evaluation, so the three second order minors of the last two columns are the only ones that need to be found. The third numerator equals

$$\frac{-x\begin{vmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 4 & -3 & -10 & -6 & -6 & 12 \\ -3 & -4 & -6 & 14 & 12 & -4 \\ 0 & 0 & 4 & -3 & -10 & -6 \\ 0 & 0 & -3 & -4 & -6 & 14 \\ 0 & 0 & 0 & 0 & 4 & -3 \end{vmatrix} + \begin{vmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 4 & -3 & -10 & -6 & -6 & 12 \\ -3 & -4 & -6 & 14 & 12 & -4 \\ 0 & 0 & 4 & -3 & -10 & -6 \\ 0 & 0 & -3 & -4 & -6 & 14 \\ 0 & 0 & 0 & 0 & -3 & -4 \end{vmatrix}}{(-25)^3}$$

$$= \frac{-14200x+13350}{(-25)^3} = \frac{568x-534}{625}.$$

In evaluating these determinants three second order minors

are new, all other factors having been previously found.

The fourth numerator equals

$$-x\begin{vmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 4 & -3 & -10 & -6 & -6 & 12 & 0 & 1 \\ -3 & -4 & -6 & 14 & 12 & -4 & 1 & -6 \\ 0 & 0 & 4 & -3 & -10 & -6 & -6 & 12 \\ 0 & 0 & -3 & -4 & -6 & 14 & 12 & -4 \\ 0 & 0 & 0 & 0 & 4 & -3 & -10 & -6 \\ 0 & 0 & 0 & 0 & -3 & -4 & -6 & 14 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & -3 \end{vmatrix} + \begin{vmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 4 & -3 & -10 & -6 & -6 & 12 & 0 & 1 \\ -3 & -4 & -6 & 14 & 12 & -4 & 1 & -6 \\ 0 & 0 & 4 & -3 & -10 & -6 & -6 & 12 \\ 0 & 0 & -3 & -4 & -6 & 14 & 12 & -4 \\ 0 & 0 & 0 & 0 & 4 & -3 & -10 & -6 \\ 0 & 0 & 0 & 0 & -3 & -4 & -6 & 14 \\ 0 & 0 & 0 & 0 & 0 & 0 & -3 & -4 \end{vmatrix}$$

$$(-25)^4$$

$$= \frac{7654x+1}{3125}$$

As in the other cases, there are but three second order minors to be found, and this rule holds for all such evaluations.

THE PARK SCHOOL, BALTIMORE, MD.

SPECIMEN EXAMINATION QUESTIONS IN JUNIOR HIGH SCHOOL ALGEBRA.

By the Mathematics Department of the Ethical Culture High School, Matilda Auerbach, *Supervisor*.

Part I. Number Systems.

I. Discuss briefly three number systems.

(*a*) State some distinguishing characteristic of each and show whether it is an advantage or disadvantage in each case.

(*b*) Mention some purposes for which the Roman numbers are used to-day.

II. State and illustrate any advantage you can think of in representing numbers by letters.

III. Illustrate some one or more advantages of extending the number system to include the negative number.

Part II. Formulas.

I. Western price list gives cost in cents per foot of sewer pipe $C \equiv 0.4d^2 + 14$, where d is diameter in inches. What would $\frac{1}{2}$ mile of 16 in. pipe cost?

II. The resistance r of a train in lbs. per ton, due to speed, is given by the formula $r \equiv 3 + (s/6)$. Construct a graph for speeds from 5 to 60 mi. per hr.

III. (*a*) Write in good English the following fact:

$$(a + b)^3 \equiv a^3 + b^3 + 3ab(a + b).$$

(*b*) Illustrate the fact stated in (*a*).

(*c*) Prove the fact stated in (*a*).

(*d*) What would be the effect on the cube of the sum of two numbers of doubling each number? Of tripling each number?

IV. If n gymnastic exercises may be taken in any order, the total number T of different ways in which they may be selected to follow each other is given by the statement:

$$T \equiv n(n-1)(n-2)\ldots 3.2.1.$$

(*a*) In how many ways may a series of 5 exercises be arranged? 6 exercises? 7 exercises?

(*b*) Using the same formula in how many different ways may 8 people be seated in a row in the theater?

V. The formula for compound interest is

$$I \equiv p\left(1+\frac{r}{100}\right)^{n}-p,$$

where I is the interest, p the principal (in dollars), r the rate and n the number of years.

(*a*) What will be the compound interest on \$700 for 3 years at 4 per cent.?

(*b*) What principal will bring \$600 interest in two years at 10 per cent. compound interest?

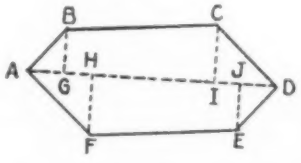
(*c*) Make a formula for the amount in a compound interest problem.

(*d*) To what will \$3,400 amount at 5 per cent. in 3 years?

(*e*) In a compound interest problem will the interest be doubled if the rate and time are unchanged but the principal doubled? Would the amount be doubled?

(*f*) Keeping the principal the same but doubling the rate (or time, keeping the other the same) would the interest (or amount) be doubled?

VI. In the polygon $ABCDEF$, $AG=a$; $GH=b$; $HI=c$; $IJ=d$; $JD=e$; $BG=f$; $HF=g$; $CI=h$; and $JE=k$. Write a formula for finding the area.



VII. Find the area of polygon in exercise VI if:

$a=100$, $c=400$, $e=150$, $g=200$,

$b=100$, $d=100$, $f=150$, $h=175$ and $k=150$.

VIII. The formula which transforms Fahrenheit readings of a thermometer into Centigrade readings is $C\equiv\frac{5}{9}(F-32)$. If $C=40°$, find the value of F.

1. Change the following readings to Fahrenheit readings:
(*a*) 0° C., (*b*) 100° C., (*c*) 50° C., (*d*) — 12° C.

2. At what temperature do the Centigrade scale and Fahrenheit scale indicate equal numbers?

3. Express F. in terms of C.

4. Represent the original formula graphically.

5. State in English the fact given in the original formula.

IX. (*a*) A cone-shaped hood for a chimney is 30 in. in diameter and 13 in. high. Allowing 1 in. on the side for locking, what surface of metal is required for the hood?

(*b*) Make a formula for use in solution of problems similar to 8.

X. Represent by a diagram:

$$(a+b)^2 \equiv a^2 + 2ab + b^2.$$

XI.
$$\begin{cases} 1+2+3+4 \ldots + n \equiv \frac{n}{2}(n+1), \\ 1+3+5+ \ldots n \text{ terms} \equiv n^2, \\ 2+4+6+ \ldots n \text{ terms} \equiv n(n+1), \\ (n+\frac{1}{2})^2 \equiv n(n+1) + \frac{1}{4}. \end{cases}$$

Using above formulas

(*a*) Find the sum of the first 600 odd numbers.

(*b*) Find the sum of the first 600 even numbers.

(*c*) Tabulate the squares of mixed numbers ending in one half from 50 to 60.

(*d*) In Italy the hours of the day are numbered from one to 24. How many strokes would a clock make per day in striking the hours?

(*e*) Find the sum of the first 600 numbers.

XII. (*a*) Find the difference between the squares of the highest and lowest of three consecutive even numbers in terms of the numbers.

(*b*) State the result as a general fact.

(*c*) Illustrate the fact.

(*d*) Is (*c*) true of odd numbers?

PART III. EQUATIONS—PROBLEMS.

I. At what advance must a shopkeeper mark goods costing 90 cts. that he may allow a 20 per cent. discount and yet gain 25 per cent.?

II. The actual cost of removing a cubic yard of rock in excavating a certain canal is \$1.10. What price should be put in the estimate, if 12 per cent. is to be allowed for superintending and 10 per cent. on the cost, including superintending, is allowed for profit?

III. At Boston and New York (225 miles apart), the prices of coal are \$6.25 and \$6.75 respectively. Coal can be shipped for one half a cent per ton per mile. Find how far from Boston it will be indifferent to a customer whether he buys coal from Boston or New York.

IV. A house valued at \$8,000 rents for \$50 per month. The repairs on the house each year amount to \$80 and the taxes are \$100. No allowance being made for change in value, does the property pay, and if so what per cent?

V. If an ore loses $41\frac{1}{2}$ per cent. of its weight in roasting, and $43\frac{3}{4}$ per cent. of the remainder in smelting, how much ore will be required to yield 1,000 tons of metal?

VI. A tower casts a shadow of 20 ft. 6 in. when a man 5 ft. 8 in. high who is passing casts a shadow of 2 ft. 9 in. Find the height of the tower.

VII. A steamboat running at the rate of 8 miles an hour sees a motor boat 10 miles off, going at the rate of 5 miles per hour. How far will the steamboat go before it overtakes the motor boat? Solve by graph.

VIII. Two vessels sailed from the same point, one north at the rate of 15 knots an hour, the other east at the rate of 20 knots an hour. How far apart were they after 6 hours?

IX. A man going from a certain place traveled at the rate of 5 miles an hour. After he had been gone 3 hours, a horseman, going at the rate of 8 miles an hour, was sent after him. How far did the latter travel before overtaking the former?

X. (*a*) A factory has a fixed charge of \$20 daily, and makes an average gross profit of \$2 daily per workman employed. What remains after the fixed charge is paid is net profit. Make a graph showing how the net profit varies as the number of workmen varies from 0 to 50.

(*b*) A second factory has a fixed charge of \$60 and makes an average gross profit of \$3 daily per workman employed. Con-

struct a graph showing the net profits of this factory in the same diagram as (*a*).

(*c*) From the graph answer the following questions:

1. For what number of workmen will the factories make the same net profit?

2. What is that profit?

3. For what number of workmen does the first factory make the larger net profit?

4. For what number of workmen does the second factory make the larger net profit?

5. What number of workmen must each employ to make $30 net profit?

6. What is the net profit of each factory if 25 workmen are employed?

PART IV. PROCESSES.

I. (*a*) State in words and algebraically one principle of addition.

(*b*) Subtract the sum of $5s^2-6s+7$ and $10-6s-4s^2$ from $s^2-6s+16$.

(*c*) If $M \equiv 2a^3-3a^2b+ab$ and $N \equiv 3a^3-2a^2b-ab$ find $MN/-a^2$.

(*d*) Simplify: $m-(n-p)+[3m-3n-6m]$.

II. Solve, stating authorities:

$$22x-[3-(3x-2)] = 2x-(4x-3)+19.$$

III. Evaluate: $16\times 3\div 2-8+5\times 2-9\div 3\times 6-12$.

IV. Divide $x^3+5x^2-8-13x$ by $x-2$.

V. (*a*) Solve graphically: $x+17y=53$ and $8x+y=19$.

(*b*) Check (*a*) by an algebraic solution.

VI. What is the coefficient of x in the reduced form of the expression

$$(x-4a)-[2a-3\{x-2(x-a)\}]?$$

VII. Simplify:

$$\frac{5}{2-a}+\frac{3}{a+2}-\frac{1}{a^2-4}.$$

VIII. Simplify:

$$\frac{x^2-5x+6}{x^2+3x+2}\cdot\frac{x^2+x-2}{x^2+x-6}+\frac{x^2-4x+3}{x^2+4x+3}.$$

IX Solve for x:

$$\frac{x}{x-3}+\frac{1}{2(x^2-x-6)}=\frac{2}{x+2}+1.$$

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JUDGING A TEACHER OF MATHEMATICS.

By William E. Breckenridge.

To the supervisor of mathematics who desires to see more in a recitation and to save time in characterizing his teachers' work, the following discussion may have some suggestions. To the young teacher of mathematics who is ambitious to make the greatest possible success, a knowledge of points on which he is to be judged may help to improve his teaching. Recently there has been great activity in making out lists of points on which a teacher should be judged. In New York City, especially, the requirement that a teacher shall be judged to be of superior merit at certain times in his career before he is advanced in salary grade, has made it necessary to define the term "a teacher of superior merit." Some of our best judgments are hardest to analyze. The late Supt. Stevens, in charge of high schools in New York City, said: "I think I know a good teacher when I visit his recitation but I cannot define him."

Undoubtedly the most important part of a teacher's work, his influence on his students, cannot be reduced to a definition. Granting this, there are, however, certain well-defined points on which a teacher may be judged.

Any plan of criticism should include strong as well as weak points. One of our best high school principals owes his popularity with his teachers to his custom of never criticizing a teacher adversely without mingling some praise with the criticism. It may be said that this is also a good plan in dealing with pupils.

The activities of a teacher are exercised in two large general fields, (A) IN THE CLASSROOM and (B) OUTSIDE OF THE CLASSROOM. The greater part of this discussion is concerned with the former. (A) *For what should a supervisor be on the alert during his visit to a classroom?*

There must be some overlapping in any classification of a

teacher's qualities, but for convenience, let us use two main heads, (I) DISCIPLINE, (II) INSTRUCTION.

(I) In judging a teacher's discipline, the supervisor should observe first of all *how a class enters the room.* Is it quiet or noisy? Experienced teachers know that students may pass from a room where they have been noisy and disorderly to a room across the hall where they will be in perfect order, because they feel that they are in a different atmosphere. Do the pupils feel when inside the door that they are on ground where it is not safe to trifle or do they feel that they are on ground which is appropriate for successful trench warfare against the teacher and begin to lay in a supply of ammunition in the shape of chalk and board erasers?

As to *whispering* or talking in the class, circumstances may make it possible for a teacher to allow a natural whispered question occasionally without disaster to the class, but usually in our large classes it is safer to allow no whispering. An annoying habit in some classes is *answering before being called by name.* This is usually due to the careless and indefinite manner in which the teacher asks questions. Care in directing every question to a particular pupil will usually remedy this defect. An effective device is calling from cards held in the hand or from a map of the room, the order of course being (1) question, (2) slight pause, (3) calling the pupil by name.

Concert answers are allowed by teachers of some subjects but they cannot be defended in a mathematical recitation.

A general *tendency to mischief or disorder* sometimes present in a class is often difficult to detect. The presence of the supervisor will usually guarantee good order while he is in the room. This particular tendency of a class can better be detected by a number of brief glances into the room.

The *attitude of the class* toward the teacher is very important. Five degrees may be distinguished:

1. The *sullen* class is "fighting mad." The teacher is holding the lid on, but by great effort. This attitude is almost always the result of injustice in the teacher's treatment of his pupils.

2. The *indifferent* class has no feeling about the teacher one way or the other, but is glad when the recitation is over.

3. In the *respectful* class, the pupils show by their acts that they hold the teacher in high esteem.

4. Between the *cordial* class and the teacher there exists real friendship, and warm mutual regard. The class welcomes and enjoys the recitation hour.

5. The *enthusiastic* class shows a love for the subject. The great truths of mathematics have taken firm root. It is sometimes difficult to keep these pupils in order because they all wish to participate in the recitation.

This subject of discipline deserves more extensive treatment than is possible within the limits of this paper. One suggestion, however, may not be amiss. In the case of an obstreperous class, it is often a decided help to the teacher for the supervisor to suggest a change in the form of the recitation. A good form for this case is where the teacher assigns a problem and the students solve it at their seats, the first one finishing going to the board and explaining. No discussion is allowed until the student at the board has finished his explanation and then only as much as the teacher thinks wise. Meanwhile the teacher is free to move about the room wherever he may be needed. The emphasis is on leading the pupil to do things rather than on restraining him from activity. New teachers finding trouble with discipline and acting on the advice of the supervisor have used this form of recitation exclusively for a week with good results in teaching and a great improvement in discipline. Of course any one form of recitation should not be continued indefinitely.

Under the head of (II) INSTRUCTION, the supervisor should observe first of all how well a class is paying *attention*. For this purpose he should take a position in the front of the room where he can see the pupil's eyes.

There are *two common causes of inattention*. The first is the *dialogue method of teaching*. After a pupil is called up, there ensues a dialogue between him and the teacher. Other pupils know they are safe as long as the pupil reciting is on his feet and they take a vacation until this pupil sits down. Then there is a moment of strained attention of the whole class, the only purpose of which is to find out who is to be called next. This is a very common fault of young teachers, especially those who

have been doing considerable tutoring. Obviously the remedy is to ask questions in such a way as to command the attention of the entire class.

Another common cause of inattention is the *failure to hold pupils immediately responsible for the teaching i. e.,* too much impression and too little expression. If the students know they are to be held responsible for the reproduction of the teaching, they will usually pay attention.

A supervisor after visiting one of his teachers remarked: "One third of your class was not paying attention and in seeking the reason, I found that you were not holding your pupils immediately responsible for the principles taught." The teacher had thought that the recitation was a good one, but after attending to the suggestion of his supervisor, he was amazed and delighted to find a marked increase in efficiency.

Is the class *interested?* If the teacher can teach the pupil thoroughly something that he did not know before, there will be a feeling of growth in power which is the greatest factor in interest. If in addition, the subject can be presented so that it seems worth while, there will be an increased interest, particularly on the part of certain types of mind. The most interesting thing to a child is the discovery of some new power in himself.

When interest lags in mathematics it is often worth while for the teacher to furnish points of contact with experience in the form of real problems or mathematical recreations. A teacher in a suburban town, finding his class in geometry dull and lifeless, applied to a supervisor for advice. The supervisor, finding that, apparently, original work had been used quite enough, suggested to the teacher that he start a discussion on the trisection of the angle, showing at the same time how to make a machine that would draw the Conchoid of Nicomedes by the aid of which any angle may be trisected. Within a week the teacher reported as follows: "The whole town is trying to trisect the angle and there is no more dullness in the mathematics class."

Again the supervisor will observe whether the class is *industrious.* Is the whole class busy all of the time? Of course it is not. But how closely does the teacher approximate to the

desired result? In this same connection, do the *pupils work rapidly or do they dawdle?* Dawdling is said to be the great fault of the American schools. A wholesome respect for time should be inculcated in the pupils and without driving them so hard as to make them nervous, there should be a reasonable speed observable in the movement of the recitation. Where more speed seems desirable, devices for obtaining it are (1) oral work with rapid fire questioning, (2) a time limit after which no work is received, (3) rating the first problem done correctly at 10, the next 9, etc., (4) the method of drill described under the discussion of Discipline.

Did the teacher *repeat pupils' answers?* His idea is that he impresses the truth upon the pupil by repeating the answer. But if the answer requires repeating, it is better to call upon another student. In the great majority of cases it is a waste of time for the teacher to repeat a pupil's answer. The students become inattentive to the recitations of their fellows and listen only when the teacher talks. The student reciting should be taught to express himself well enough to be heard by everyone in the room.

Was the *result of the hour's work* a reasonable degree of love for the subject and new power on the part of the pupils?

Some supervisors use this as their only test. They say: "The proof of the pudding is in the eating. It does not matter how it is made."

A great many factors make up *personality,* but the supervisor will ask at least the following questions:

Is the teacher sufficiently *forceful?* Force does not mean loud talking. It does mean that the teacher must have energy enough so that his teaching will carry to every student in the room.

Is the teacher *poised or nervous?*

Is he *alert?* The writer knows a teacher who has an eye like an eagle and who sees things in his class long before they occur. In other words the teacher notes instantly the slightest inattention and is ready with the remedy before the fault would be noticed by a visitor to the class. It is the business of every instructor to watch his class and know just what everyone is doing at every moment. A young teacher especially needs to be cautioned on this point. He should not turn his back on his class until he is sure of control. If there is to be writing

on the blackboard, let a pupil do it. The teacher should take such a position that he can watch the class and teach at the same time.

Is the teacher *enthusiastic?* Enthusiasm may be natural, cultivated, or assumed, but it is vital to the best teaching. To him who is inspired with the message of the great truths of mathematics and who is in sympathy with the life of boys and girls enthusiasm should be natural and contagious. One of the best teachers that the writer has ever known was compelled to assume the enthusiasm which she knew was essential to success. When complimented on a recitation and especially on her great enthusiasm for the subject, she replied "Do you know I hate the whole thing?" Although this is an extreme instance, the success of the teacher in cultivating and assuming enthusiasm may be comforting to some of us in a similar position.

Is the teacher *sympathetic* toward the weak but earnest students? Some of the oldest members of our profession are lacking in the respect. They have lost their sympathy for boys and girls and their appreciation of the difficulties that beset the mind of the weak pupil in his mathematics lesson. A real missionary spirit is desirable for this kind of work.

On the other hand, is the teacher *quick to detect and adequately deal* with the lazy or dishonest student? It is rare that the same teacher is good in this respect and at the same time is sympathetic.

Does the teacher *win cooperation*, or does he antagonize his students?

Neatness in dress deserves only this comment: It is one thing that a supervisor can write down in cold black and white as a fact. An examiner of teachers recently said "It is bad enough to say to a man that he is lacking in neatness, but to say this to a woman is about as much as my life is worth."

Again, the supervisor will ask "Is the teacher *dignified in bearing* and position in the classroom?" A teacher during his second year out of college was giving a lesson to a class of boys in the presence of a visiting superintendent whose school the teacher desired to enter. During the entire recitation, the teacher was tipped back in his chair with his feet on the desk. At the close of the hour, the superintendent complimented him on a fine recitation, but added: "If you come with us, you will

have to get your feet down from the desk. We have girls in our school." Now it had never before occurred to the teacher accustomed to the easy, lounging attitudes of college life that there was anything undesirable in his attitude. This matter of dignity in position and bearing is therefore commended especially to young teachers. Is the *voice* clear, distinct, pleasing, and well modulated? The late Rev. T. T. Munger, of New Haven, when asked how he would tell whether a man would make a good public speaker, replied: "I would judge him as I would judge a horse—look in his mouth." Certainly a prospective teacher should be sure that he has a good voice or he will be most seriously handicapped.

In the *use of English* are any defects apparent?

Is the teacher *courteous* to the class? Because he has absolute authority is no reason why he should treat his pupils with sarcasm and ridicule. The writer has in mind a boy who begged with tears running down his cheeks to be transferred out of the class of a teacher who, the boy said, "made a fool of me before the class."

Has the teacher *self control?* The days when the supervisor came into the room and intentionally stumbled into the waste basket in order to test the self control of both teacher and pupils are happily gone. There are plenty of situations developing naturally that sufficiently test self control.

Has he *good health?* By one of our foremost educators this matter of health is placed first of all qualities that make up the equipment of a teacher.

Is he *resourceful or dependent?* Is it true of him as one boy said: "My teacher has the advantage of me in geometry, because he has the book open?" In power of illustration, is the teacher able to draw from a large stock of material?

These are the principal points under personality.

Largely dependent upon personality, is a teacher's *skill in teaching.* Under this head a supervisor would ask:

Is there *motivation?* Did the teacher make the subject seem worth while to the student? Few take time at the beginning of the recitation to show the motive of the work. Where there is strong motivation, there is a great increase in interest and efficiency. There is no part of algebra or geometry where the

teaching may not be strengthened by a few minutes devoted to this important part of instruction.

Is the teaching *scholarly?* At the close of the teaching has the theory of the topic been exhausted? Scholarly teaching does not necessarily mean bringing in higher mathematics, but it does involve a wider view of the subject than can be gained from a single text book.

Is the arrangement of the lesson *logical?* Much time is often lost by poor arrangement.

Is the teacher clever in *questioning?*

In this connection, the writer strongly recommends the article on Questioning by Prof. Stevens, which may be obtained from The Bureau of Publications of Teachers College. Under this subject the supervisor will note how many natural questions were asked by the class, *i. e.*—questions relevant to the subject asked because the pupils desired information. Of the questions asked by the teacher, how many stimulated thought in the pupils? How many were questions of fact? Were the questions well distributed about the class? Too often the teacher is tempted to call only upon those whose hands are raised.

A good teacher will watch for the inattentive student and call upon him in order to command his attention. The weak students will not be neglected, but will receive the greater portion of the time of the recitation. In this connection, the writer is tempted to repeat a story told at a recent meeting of mathematics teachers by Mr. Albert H. Wheeler. A superintendent, who was visiting a teacher's class, observed that he liked to see pupils raise their hands. It showed, he said, that they were interested in the subject. After the departure of the superintendent the teacher instructed his class as follows: "If you know the answer to my question, raise your right hand, if you do not know it, raise your left." When the Superintendent called again, the result can be imagined.

Again, is there evidence that the *work is planned?*

Every teacher should have a plan for the term and a detailed plan for every recitation carefully prepared in advance. Always it is well to plan the work. Generally it is well to work the plan, the only exception being in case the prepared lesson does not seem adequate for the occasion.

Is there a correct ratio between the teaching and drill?

Does the teacher have any method of finding out whether every pupil has studied his lesson adequately?

If so, *does he follow up delinquents?* These two points are very important, for upon them depend the development of industry in the pupils. A teacher should find out before a pupil has left the room whether he has adequately prepared his lesson. One good way to accomplish this is to give a brief test, say of five minutes, at the beginning of every recitation. In geometry the test question must be given from the advance lesson. If the proposition is too long for five minutes, require only an outline proof which means the steps in order without the reasons. By assigning a captain to every row these tests can be corrected and marked roughly c = correct or nearly so, h = half correct, w = all wrong. While the recitation is going on, these tests may be corrected and the results entered in the teacher's record book by a pupil acting as secretary. Hence it is possible, even in geometry, within twenty minutes after the recitation begins, to find out who has studied his lesson adequately. There is a great advantage in getting hold of the boy who has neglected his lesson before he leaves your recitation.

But it will be futile to find out who is delinquent unless he is followed up. A school that does excellent work on examinations gives as the secret of its success the simple fact that if a boy fails in a lesson he stays after school till he makes it up and that this is carried on persistently day after day. Possibly some of us will not believe in carrying the nursing process to that extent, but it surely is productive of excellent results on examinations.

The *home work* should be inspected by the supervisor and judged as to quantity, quality, and form.

Did the teacher show *power to use devices for expediting class work?* For example if there was paper given out, was it done in an orderly and expeditious manner?

Were the pupils *taught how to study the next lesson?*

Were they required to maintain *good posture* both sitting and standing?

These are the important things in the application of the teacher's skill to his work and complete the points on which to judge a teacher within the classroom.

(*To be concluded in the next issue.*)

NEW BOOKS.

The Development of Arabic Numerals in Europe. By G. F. Hill. Oxford: the Clarendon Press. Pp. 125. — 7/6.

The author of this work is the curator of the department of coins and medals in the British Museum and the date of an Italian medal of the Sultan Mahomet II being called in question, he set about the task of collecting and classifying as many examples as possible of securely dated instances of the Arabic numerals.

Some sixty-four tables are given from which may be traced the development in Europe of our present forms of numerals. In examining these tables the reader will find some surprising features especially in the development of 2, 4, and 7, and much interesting information. It is a careful and scholarly piece of research.

Man's Place in Nature and Other Essays. By Thomas Henry Huxley. New York: E. P. Dutton & Co. Pp. 372.

Those who are not familiar with *Everyman's Library*, edited by Ernest Rhys, of which this is a volume, will do well to examine it, for they will find a long list of standard works on a wide variety of subjects. The volumes are well edited, well made, and published at a price that is surprisingly low for the quality they contain.

Selections from Carlyle. Edited by Samuel B. Hemmingway and Charles Seymour. Boston: D. C. Heath & Company. Pp. 260. 75 cents.

This is an attractive volume of carefully chosen selections from Sartor Resartus,The French Revolution and Past and Present. The book is intended for college use and is furnished with numerous explanatory notes, placed conveniently at the foot of the page beneath the text, which furnish a historical background, should stimulate the student's interest.

The Adolescent Period, Its Features and Management. By Louis Starr. Philadelphia: P. Blakiston's Son & Co. Pp. 211. $1.00 net.

The object of this book is to present an outline of the physical and psychical changes that are to be expected in the period of adolescence and also to point out methods of management to overcome the difficulties and dangers of this trying period. While the author is a physician he writes in a non-technical style easily understood by the general reader. Teachers would do well to learn more of the physical and psychical bases for educational work and this book should be a great help to that end.

The Meaning of Education. By NICHOLAS MURRAY BUTLER. Revised and enlarged edition. New York: Charles Scribner's Sons. Pp. 385. $1.50 net.

This new and enlarged edition of President Butler's book will be welcome to all who are in any way interested in the fundamental principles on which educational practice should be based. Two chapters of the former edition, viz., those on "Democracy and Education" and "The Reform of Secondary Education in the United States" have been omitted and some fourteen new chapters on live educational topics take their place.

Dr. Butler's standing in the educational world gives his writings an authority which is well worthy of the respect of every teacher. We predict for it a cordial reception.

Robert of Chester's Latin Translation of the Algebra of Al-Khowarizmi with an Introduction, Critical Notes and an English Version. By LOUIS CHARLES KARPINSKI. New York: The Macmillan Co. Pp. 164. $2.00.

The real services of the Arabs to science is not very generally known and this translation will help teachers of mathematics and others to see more clearly that they were not only the medium through which Hindu and Greek mathematical knowledge came to Europe after the Middle Ages, but they made contributions of their own. It is a scholarly piece of work on the part of Dr. Karpinski and printed in fine form.

Historical Introduction to Mathematical Literature. By G. A. MILLER. New York: The Macmillan Co. Pp. 302.

The main object of this work is to lead the student to points where he may overlook domains of considerable extent in order that he may be able to get a comprehensive view and perhaps a desire to cultivate some of the regions more fully. The reader will secure a much better appreciation of the rôle played by mathematics in the civilization of the present day. There are chapters on: Mathematical literature, historical questions relevant to mathematics, fundamental developments in arithmetic, in algebra, and in geometry, twenty-five deceased mathematicians, lists of important works. The author has produced a very excellent work which will do much to give those interested in general education, as well as students and teachers of mathematics, a clear and comprehensive view of the field.

The Pioneer Boys of the Yellowstone. By HARRISON ADAMS. Boston: The Page Company. Pp. 346. $1.25.

This is a description of the adventures of the Pioneer Boys with the Lewis and Clark Expedition. They became separated from the rest of the party and lost in Yellowstone Park. Every boy who is fond of adventure and stirring scenes will find this a very entertaining book. It is founded on facts and true to the times and circumstances.

Alfred the Great. By BEATRICE ADELAIDE LEES. New York: G. P. Putnam's Sons. Pp. 493. $2.50.

There are many biographies of King Alfred the Great, and the excuse for another is, in the words of the author, "The abiding charm and permanent importance of the subject." It is not a mere covering of the same ground that others have used in their biographies, but in addition Miss Lees has taken the results of recent investigations, which have thrown much light on obscure points, and brought them together in a compact form suitable for the general reader. It is an exceedingly well written book and well worth reading, even if we have already read others. It is well and profusely illustrated and a splendid piece of bookmaking.

High School and Class Management. By HORACE A. HOLLISTER. Boston: D. C. Heath & Co. Pp. 314.

The author's experience as high school visitor at the University of Illinois has well fitted him for a work of this kind. The material is all first-hand and the discussions the results of his experiences. It is carefully written and teachers of mathematics will do well to read it.

Mortality Laws and Statistics. By ROBERT HENDERSON. New York: John Wiley and Sons. Pp. v + 111.

In this book Mr. Henderson takes up something of the historical development of the subject, devotes considerable attention to its mathematical discussion, and then considers the mortality tables. The book does not, however, consider monetary applications.

It is concisely and clearly written, and should prove a contribution to actuarial literature.

First Year Mathematics for Secondary Schools. By ERNST R. BRESLICH. Chicago: The University of Chicago Press. Pp. xxiv + 344. Price $1.00 + postage.

This is one of the University of Chicago Mathematical Series, and is a development from the earlier books of the same title by George W. Myers. The University High School has been experimenting with correlated mathematics since 1903, and the mathematical faculties of the University School, the School of Education and the University have worked together to help Mr. Breslich to produce a first year text on mathematics, instead of on one of its branches. The author seems to have succeeded in producing a book more coherent than its predecessors in this field. There is less feeling that one has barely gotten a glimpse of algebra before being whisked over into geometry, or vice versa, and a more evident reason for the placing of different parts in the order given them.

For classes that will have but one year of high school mathematics there is strong reason for giving this book serious consideration, for it

should equip the pupils with some knowledge of all the elementary fields. Whether it is also desirable for pupils who will take two or more years of mathematics is an open question.

Whatever a teacher's feeling about the close correlation of the mathematical subjects, this book will prove worthy of study. It has many interesting features in its details, as well as in the general question of its method.

Plane and Spherical Trigonometry and Tables. By GEORGE WENTWORTH and DAVID EUGENE SMITH. Boston: Ginn & Co. Pp. 104 + 230. Price $1.35.

This book differs from its predecessors in the series principally in the order of topics. Application of right triangle methods to problems that involve only small numbers is immediately followed by the introduction of logarithms and by the solution of oblique triangles. Identities, trigonometric equations, graphs, etc., are left until the end of plane trigonometry. There is also a chapter on the Applications of Trigonometry to Algebra, including De Mowre's Theorem, Euler's Formula and the natural system of logarithms. The spherical chapters contain the usual material.

The lists of problems are excellent and the text seems to be interesting and clear. The tables are unusually readable, as great care has been taken to make the pages less crowded.

Every Man a King. By ORISON SWIFT MARDEN. New York: Thomas Y. Crowell Company. Pp. 240. $1.00 net.

Dr. Marden's books are always good and this volume is no exception. Its sound reasoning and clear optimism are fascinating and inspiring, and interest begins with the first chapter. The reading of this book would mean success to many young people who without it are headed for failure. It should be in every library and in constant use.

Acres of Diamonds. By RUSSELL H. CONWELL. New York: Harper & Brothers. Pp. 181. $1.00 net.

The major portion of this volume is taken up with the story of the life and achievements of Dr. Conwell, by Robert Shackleton. The rest is the story of a lecture, delivered at various places, putting forth the idea that every one has "the opportunity to make more of himself than he does in his own environment with his own skill, with his own energy, and with his own friends."

It is tremendously interesting and should prove a great source of inspiration and help to young and old alike; and shape their lives for success and happiness.

Heart of Europe. By RALPH ADAMS CRAM. New York: Charles Scribner's Sons. Pp. 325. $2.50 net.

That part of Europe between the Seine and the Rhine, celebrated for its history and monuments of artistic beauty, has been so destroyed and desecrated by the present unnecessary and cruel war that the author cries out with a broken heart against this so-called "highest civilization in Europe."

Mr. Cram gives a vivid account of the evolution of Belgium and a powerful arraignment of the destruction of its magnificent schools and churches. His artistic, poetic, and religious nature causes him to write in a vivid, entertaining and instructive style that carries the reader with him. The book is beautifully illustrated.

The Boy's Life of Lord Roberts. By HAROLD F. B. WHEELER. New York: Thomas Y. Crowell Company. Pp. 272. $1.50.

Perhaps the best as well as the most interesting history for boys is the biography of some strong, noble man or woman who have served their country with sincerity and with humility. Such is that of Lord Roberts. He was a man of keen insight, singleness of aim and genuineness of purpose, who was beloved by all and a national hero because of his character coupled with his deeds.

Teachers will do well to encourage every boy to read this book.

College Sons and College Fathers. By HENRY SEIDEL CANBY. New York: Harper & Brothers. Pp. 233. $1.20 net.

This is an examination of college education from three standpoints—that of the student, that of the professor, and that of the results achieved. The author has a keen insight into college life and problems, and while it is written primarily for students and parents, college and school teachers will find it very suggestive and helpful. It contains much of sound educational wisdom and to-day when there are so many influences tending towards a narrow vocational training it is encouraging to find a clear note sounded in favor of an irreducible minimum of liberal education. "If we exclude or unduly delimit a liberal training in our colleges, and especially in our schools, as sure as night follows day there will be a decrease, and a sharp one, in the intellectual sympathy which makes intellectual leadership possible. Cut out history, cut out literature, cut out mathematics beyond its elements, and in a stroke you cut three of the bonds that unite society."

NOTES AND NEWS.

On December 30 and 31, 1915, there was held at Columbus, Ohio, the organization meeting of a new national mathematical association, the call for which had been signed by 450 persons representing every state in the Union, the District of Columbia, and Canada. The object of the new Association is to assist in promoting the interests of mathematics in America, especially in the collegiate field. It is not intended to be a rival of any existing organization, but rather to supplement the Secondary Associations on the one hand, and the American Mathematical Society on the other, the former being well organized and effective in their field, and the latter having definitely limited itself to the field of scientific research. In the field of collegiate mathematics, however, there has been, up to this time, no organization and no medium of communication among the teachers, except the *American Mathematical Monthly*, which for the past three years has been devoted to this cause. The new organization, which has been named the Mathematical Association of America, has taken over the *Monthly* as its official journal.

There were 104 persons present at the organization meeting. The constitution and by-laws together with a full report of the proceedings will be published in the January issue of the *Monthly*. The following officers were elected:

President, Professor E. R. Hedrick, University of Missouri.

First Vice-President, Professor E. V. Huntington, Harvard University.

Second Vice-President, Professor G. A. Miller, University of Illinois.

Secretary-Treasurer, Professor W. D. Cairns, Oberlin College.

Publication Committee, Professor H. E. Slaught, University of Chicago, Managing Editor, Professor W. H. Bussey, University of Minnesota, and Professor R. D. Carmichael, University of Illinois.

These officers, together with the following, constitute the *Executive Council:*

Professor R. C. Archibald, Brown University.
Professor Florian Cajori, Colorado College.
Professor B. F. Finkel, Drury College.
Professor D. N. Lehmer, University of California.
Professor E. H. Moore, University of Chicago.
Professor R. E. Moritz, University of Washington.
Professor M. B. Porter, University of Texas.
Professor K. D. Swartzel, Ohio State University.
Professor J. N. Van der Vries, University of Kansas.
Professor Oswald Veblen, Princeton University.
Professor J. W. Young, Dartmouth College.
Professor Alexander Ziwet, University of Michigan.

THE ASSOCIATION OF TEACHERS OF MATHEMATICS IN THE MIDDLE STATES AND MARYLAND

THE Twenty-fifth meeting of the association was held in Drexel Institute, Philadelphia, November 27, 1915. Morning session called to order 10:30 in the Chemical Lecture Room. At this session two very interesting papers were read: "Mathematics and Psychology," by Charles C. Groves, Columbia University, New York, and "Opportunity for School Men in Public Service," by William H. Maltbie, director of the Bureau of State Municipal Research, Baltimore. The nomination committee then handed in the following report: For President, Jonathan T. Rorer, Wm. Penn High School, Philadelphia; Vice-President, Elizabeth B. Cowley, Vassar College; Secretary, F. Eugene Seymour, State Normal School, Trenton, N. J.; Treasurer, Edward D. Fitch, Episcopal Academy, Philadelphia; Member at large of the Council, Eugene R. Smith, The Park School, Baltimore, Md. Upon motion the report of the committee was accepted and the persons elected.

THE revised Constitution and By-Laws as reported in the TEACHER, June, 1915, were unanimously adopted.

Article IV of the new constitution provides that each section shall elect one representative to the Council. The sections of our association are requested to hold this election at the first opportunity, and to inform the secretary of their chosen representative.

The spring meeting of the association will probably be held in New York City during the early part of May. The spring vaca-

tions of the many schools occur either before or after Easter; this makes it inconvenient for teachers to attend the meetings of the association in April, so that the council has wisely postponed the meeting until May. An important feature of the morning program will be a survey of the present situation resulting from the criticism of mathematics teaching. It is hoped to obtain reports from as many sections of our territory as possible, which will state to what extent school and college courses have been modified.

The afternoon meeting was divided into two sections—one for the discussion of Secondary School Mathematics, where the subject "College Preparation," was discussed, with the following speakers acting as leaders: Fletcher Durell, Lawrenceville School; Harry English, High School, Washington, D. C.; Cecil A. Ewing, The Tome School; Amelia C. Wight, High School for Girls, Philadelphia; Romeyn H. Rivenburg, Peddie Institute; C. A. Bergstresser, Boys' High School, Brooklyn. In the other section for the discussion of college mathematics the subject, "Freshman Mathematics," was discussed with the following speakers acting as leaders: H. L. Hodgkins, George Washington University; Abraham Cohen, Johns Hopkins University; Clara L. Bacon, Goucher College; Charles C. Grove, Columbia University; John A. Miller, Swarthmore College; George H. Hallett, University of Pennsylvania.

Mathematical Teachers in New England.

The thirteenth annual meeting of The Association of Mathematical Teachers in New England was held on Saturday, December 4, in Boston. The morning session was held at the Boston University and the afternoon session at the Massachusetts Institute of Technology.

The meeting was called to order by President Coolidge, and the ballot for officers for 1916 resulted as follows: *President*, Professor Julian L. Coolidge, Harvard University; *Vice-President*, Professor Frank C. Moore, New Hampshire College; *Treasurer*, Frederick W. Gentleman, Mechanic Arts High School, Boston; *Members of the Council*, Miss Clara A. Snell, High School, Melrose, Mass.; Mr. A. Harry Wheeler, High School, Worcester, Mass. The treasurer's report was read, accepted, and placed on file.

An amendment to the constitution which was offered at the spring meeting on May 1, was voted on and passed. The amendment reads as follows: "Resolved that Section 1 of the constitution be amended so as to read, This society shall be called The Association of Teachers of Mathematics in New England."

The speakers for the morning session were Mr. Milton E. MacGregor, Mechanic Arts High School, Boston, "The Teaching of Quadratic Equations"; Professor Helen A. Merrill, Wellesley College, "The Napier Tercentenary"; Mr. Francis A. Rugg, Berkeley Preparatory School, Boston, "First Weeks in Plane Geometry."

At the close of an extended discussion of these papers, luncheon was served to about fifty at the Technology Union.

In the afternoon, the Committee on the Status and Welfare of Mathematics in the Secondary Schools presented its second report. Professor Tyler, chairman of the committee, outlined the work which the committee has planned for completing its investigations. Other members of the committee presented papers which will be embodied later in the printed report. These papers were: "Preparation of Teachers of Mathematics in Massachusetts High Schools," Dean F. C. Ferry, Williams College; "The Status of Algebra as a Required Subject in Massachusetts High Schools," Principal F. P. Morse, Revere High School; "A List of Topics in Algebra," Headmaster G. W. Evans, Charlestown High School.

A MAN with eight years' experience in secondary and college teaching wishes a college position. He is a Ph.D. from The Johns Hopkins University, and can give excellent references. His work is known to some of the editors of this magazine, and his name and further details will be given to any institution on request.